Perturbation Theory notes

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(Dated: April 21, 2015)

I. PERTURBATION THEORY GENERAL REMARKS

There are not many differential equations that can be solved exactly. This is especially the case for non-linear equations. If the differential equation has a small nonlinear term that prevents us from obtaining the exact solution, then often we can use the exact solution of the differential equation without the small nonlinear term and use perturbation theory to get a series solution corrections to the actual differential equation. Suppose the class of differential equations can be put in this form,

$$\ddot{x} + \omega^2 x + \epsilon\, h\left(\dot{x}, x\right) = 0\,, \qquad 0 < \epsilon \ll 1 \tag{1}$$

where the derivatives are with respect to time say. Here the ***unperturbed*** part is $\ddot{x} + \omega^2 x = 0$, which is the differential equation for a simple harmonic oscillator. Its solutions are $x(t) = A\,\cos(\omega t) + B\,\sin(\omega t)$, where A and B are constants that get determined by the initial conditions and ω is real and denotes the angular frequency. One expects the solutions to Eq. 1 to be small modifications of $x(t) = A\,\cos(\omega t) + B\,\sin(\omega t)$.

Perturbation theory allows us to find refinements to this solution systematically and in terms of ***powers*** of ϵ.

In general one does so by assuming that the solution to Eq. 1, that we seek can be written in the form

$$x(t) = x_0(t) + \epsilon\, x_1(t) + \epsilon^2 x_2(t) + \cdots \tag{2}$$

where $x_0(t)$, $x_1(t)$, $x_2(t), \ldots$ are functions to be found. Note that, because of the higher powers of ϵ, later terms make smaller and smaller contributions to $x(t)$. Let us demonstrate it through examples. Below we suppress the dependence on t for simplicity, i.e., we write x for $x(t)$.

II. EXAMPLE: STANDARD PERTURBATION THEORY

Use perturbation theory and ignore $\mathcal{O}(\epsilon^2)$ terms to solve

$$\begin{aligned} \dot{x} + x &= \epsilon x^2 \\ x(0) &= 1 \quad . \end{aligned} \tag{3}$$

Plug in Eq. 2 to get

$$\begin{aligned} \left(\dot{x}_0 + \epsilon \dot{x}_1 + \epsilon^2 \dot{x}_2 + \cdots\right) + \left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\right) - \epsilon\, \left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\right)^2 &= 0. \\ x_0(0) + \epsilon x_1(0) + \epsilon^2 x_2(0) + \cdots &= 1 \quad . \end{aligned}$$

We group terms according to the power of ϵ that premultiplies.

$$\begin{aligned} \dot{x}_0 + x_0 &= 0 \\ \dot{x}_1 + x_1 &= x_0^2 \\ \dot{x}_2 + x_2 &= (2x_0 x_1) \\ &\vdots \end{aligned}$$

I am using $\epsilon\,\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots\right)^2 = \epsilon\,\left(x_0^2 + \epsilon 2 x_0 x_1 + \mathcal{O}(\epsilon^2)\right)$. Solving the equations one by one now. The first equation can be solved by separation of variables. $\dot{x}_0 + x_0 = 0 \implies \frac{dx_0}{x_0} = -dt$, which gives $x_0 = ce^{-t}$. But the

initial condition has to hold for all ϵ so $x_0(0) = 1$, we have then $c = 1$ and $x_0 = e^{-t}$. Next we solve the second equation (i.e., $\mathcal{O}(\epsilon)$). Using what we just found, the equation becomes $\dot{x}_1 + x_1 = e^{-2t}$. The homogeneous part is $\dot{x}_1^h + x_1^h = 0$ whose solution is $x_1^h = e^{-t}$ (note that we don't introduce a constant). To find a particular solution we guess $x_1^p = Ae^{-2t}$, then plugging it in we have $-2Ae^{-2t} + Ae^{-2t} = e^{-2t}$, canceling e^{-2t} we have $A = -1$. Hence $x_1 = x_1^h + x_1^p = e^{-t} - e^{-2t}$. So the final solution is

$$\begin{aligned} x(t) &= x_0 + \epsilon x_1 + \mathcal{O}\left(\epsilon^2\right) \\ &= e^{-t} + \epsilon\left(e^{-t} - e^{-2t}\right) + \mathcal{O}\left(\epsilon^2\right). \end{aligned}$$

III. EXAMPLE: TWO TIME-SCALE

First find the exact solution then the lowest-order approximate solution to :

$$\begin{aligned} \ddot{x} + x + \epsilon\left(\dot{x}\right) &= 0 \\ x(0) = 0 \quad , \quad \dot{x}(0) = 1 \quad . \end{aligned} \tag{4}$$

If you compare this with Eq. 1, you see that $h\left(x, \dot{x}\right) = \dot{x}$ and $\omega^2 = 1$.

The exact solution: note that it is a linear ode so we can just solve it. The characteristic equation is $r^2 + \epsilon r + 1 = 0$ so $r = \frac{-\epsilon \pm \sqrt{\epsilon^2 - 4}}{2} = -\frac{\epsilon}{2} \pm \frac{i}{2}\sqrt{4 - \epsilon^2}$. So we have

$$x(t) = e^{-\frac{\epsilon t}{2}}\left\{Ae^{\frac{it}{2}\sqrt{4-\epsilon^2}} + Be^{-\frac{it}{2}\sqrt{4-\epsilon^2}}\right\}.$$

Using the initial conditions $x(0) = A + B = 0$ we find that $A = -B$. So $x(t) = Ae^{-\frac{\epsilon t}{2}}\left\{e^{\frac{it}{2}\sqrt{4-\epsilon^2}} - e^{-\frac{it}{2}\sqrt{4-\epsilon^2}}\right\} = 2iAe^{-\frac{\epsilon t}{2}}\sin\left[\frac{t}{2}\sqrt{4-\epsilon^2}\right]$. To use the second initial condition we calculate $\dot{x} = 2iAe^{-\frac{\epsilon t}{2}}\left(-\frac{\epsilon}{2}\right)\sin\left[\frac{t}{2}\sqrt{4-\epsilon^2}\right] + 2iAe^{-\frac{\epsilon t}{2}}\left(\frac{1}{2}\sqrt{4-\epsilon^2}\right)\cos\left[\frac{t}{2}\sqrt{4-\epsilon^2}\right]$, we now have $\dot{x}(0) = iA\left(\sqrt{4-\epsilon^2}\right) = 1$, which can be solved for A to give

$$x(t) = \frac{2e^{-\frac{\epsilon t}{2}}}{\sqrt{4-\epsilon^2}}\sin\left[\frac{t}{2}\sqrt{4-\epsilon^2}\right] \qquad \text{exact.} \tag{5}$$

Now we go after the perturbation theory approximation. Why does the standard perturbation theory fail? If we plug in for x, the expansion in Eq. 2, then we get

$$\left(\ddot{x}_0 + \epsilon\,\ddot{x}_1 + \cdots\right) + \left(x_0 + \epsilon\,x_1 + \cdots\right) + \epsilon\left[\left(\dot{x}_0 + \epsilon\,\dot{x}_1 + \cdots\right)\right] = 0$$

Organizing the terms in powers of ϵ we obtain

$$\begin{array}{cc} \mathcal{O}\left(\epsilon^0\right) & \ddot{x}_0 + x_0 = 0 \\ \mathcal{O}(\epsilon) & \ddot{x}_1 + x_1 = -\dot{x}_0 \\ \vdots & \vdots \end{array}$$

Anticipating solving for x_0 first and then using it to solve x_1, in the second equation, I brought all the "known" x_0 dependences to the right hand side. $\mathcal{O}\left(\epsilon^0\right)$ part clearly yields $x_0 = A\ \sin t + B\ \cos t$. Looking at $\mathcal{O}(\epsilon)$, we see that the homogeneous part of the solution, i.e., $\ddot{x}_1 + x_1 = 0$ has the same form of solution as x_0. But $\dot{x}_0$ will inevitably involve $\sin t$ and $\cos t$. When the homogeneous part has the same sinusoidal dependence as the forcing term , then there will be secular terms (a.k.a. resonance or simply disaster). In this case if we insist we will find that the corrections will not stay small for large t and perturbation theory breaks down. This is often the case for $h\left(x, \dot{x}\right)$

that involve $x$or $\dot{x}$. So the remedy is:

Introduce two time scales and think of them as two independent variables defined by

$$\tau \equiv t$$
$$T \equiv \epsilon t$$

and use the expansion (compare with Eq. 2)

$$x(\tau, T) = x_0(\tau, T) + \epsilon\, x_1(\tau, T) + \epsilon^2\, x_2(\tau, T) + \cdots \tag{6}$$

Our next step is always to plug this into the differential equation (Eq. 4) but we need to recast the differentiation in terms of t into differentiation in terms of τ and T. This is a simple application of the chain rule and is in your book (and lecture notes). They are simple to do. Use the following:

$$\dot{x} = \partial_\tau x_0 + \epsilon\left(\partial_T x_0 + \partial_\tau x_1\right) + \mathcal{O}\left(\epsilon^2\right) \tag{7}$$
$$\ddot{x} = \partial_{\tau\tau} x_0 + \epsilon\left(\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0\right) + \mathcal{O}\left(\epsilon^2\right) \tag{8}$$

where in this notation we mean for example $\partial_\tau x_0 = \frac{\partial x_0}{\partial \tau}$, $\partial_{\tau\tau} x_0 = \frac{\partial^2 x_0}{\partial \tau^2}$ and $\partial_{\tau T} x_0 = \frac{\partial^2 x_0}{\partial \tau \partial T} = \frac{\partial^2 x_0}{\partial T \partial \tau} = \partial_{T\tau} x_0$, i.e., the order of partial differentiation doesn't matter. With these preliminaries we are ready to attack the problem. As always, plug in and organize terms according to powers of ϵ– We plug in Eqs. 6-8 into Eq. 4 and collect terms. We suppress the dependence on τ and T for notational simplicity

$$\begin{aligned} \ddot{x} + x + \epsilon\dot{x} &= \\ \left[\partial_{\tau\tau} x_0 + \epsilon\left(\partial_{\tau\tau} x_1 + 2\partial_{T\tau} x_0\right)\right] + \left(x_0 + \epsilon\, x_1\right) + \epsilon\left(\partial_\tau x_0\right) + \mathcal{O}\left(\epsilon^2\right) &= 0 \end{aligned}$$

Let us group terms as before in order

$$\begin{array}{cc} \mathcal{O}\left(\epsilon^0\right) & \partial_{\tau\tau} x_0 + x_0 = 0 \\ \mathcal{O}(\epsilon) & \partial_{\tau\tau} x_1 + x_1 = -2\partial_{T\tau} x_0 - \partial_\tau x_0 \\ \vdots & \vdots \end{array}$$

where again, anticipating first solving x_0 and then x_1, in the second equation we brought all the x_0 dependence to the right. Solving the $\mathcal{O}(1)$ term we get

$$x_0 = A(T)\sin\tau + B(T)\cos\tau \tag{9}$$

Here A and B are constants with respect to τ but can in principle depend on T. Since we want the lowest order solution, it is sufficient to find A and B. To do so we force the secular term on the right hand side of $\mathcal{O}(\epsilon)$ term to zero. This term is

$$\begin{aligned} -2\partial_{T\tau} x_0 - \partial_\tau x_0 &= -2\left(A'\cos\tau - B'\sin\tau\right) - \left(A\cos\tau - B\sin\tau\right) = \\ &= \left(-2A' - A\right)\cos\tau + \left(2B' + B\right)\sin\tau \end{aligned}$$

where $A' = \frac{dA}{dT}$ and $B' = \frac{dB}{dT}$.

To make this zero we demand $2A' + A = 0$ and $2B' + B = 0$. These are easy to solve. For example $2A' + A = 0$ implies

$$2\frac{dA}{dT} = -A \implies 2\frac{dA}{A} = -dT$$

integrating both sides we get $2\ln A = -T + const.$, we have $A = A_0 e^{-T/2}$. Similarly $B = B_0 e^{-T/2}$. Plugging all of this into Eq. 9 we find

$$\begin{aligned} x &= e^{-T/2}\left(A_0\sin\tau + B_0\cos\tau\right) \\ &= e^{-\epsilon t/2}\left(A_0\sin t + B_0\cos t\right) \end{aligned}$$

We can now use the initial conditions. Note that $t = 0$ implies $\tau = T = 0$. $x(0) = B_0 = 0$ and $\dot{x} = e^{-\epsilon t/2}(A_0 \sin t)\left(-\frac{\epsilon}{2}\right) + e^{-\epsilon t/2}(A_0 \cos t)$ and using i.c. $\dot{x}(0) = A_0 = 1$. Our final solution is

$$x = e^{-\epsilon t/2} \sin t$$

If you compare this with the exact solution (Eq. 5) you find that they are equal if you ignore (i.e., set to zero) the ϵ^2 in the exact solution.

A Final note: setting the secular terms equal to zero requires grouping them up and is not always easy. I hope these notes help .